

An elementary proof of the non-renormalization theorem for the Wess-Zumino model

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Using the exact renormalization group (ERG) differential equation, we give an elementary proof of the non-renormalization theorem for the Wess-Zumino model. We introduce auxiliary fields to linearize the supersymmetry transformation, but we do not rely on the superfield techniques. We give sufficient background material on the Wilson action and the ERG formalism to make the paper self-contained.

§1. Introduction

The purpose of this paper is to prove the non-renormalization theorem for the Wess-Zumino model¹⁾ using an elementary method. Here, what we call the non-renormalization theorem is the absence of radiative corrections to the superpotential. Two of its important implications are that the expectation value of the scalar field is the same as at the tree level, and that the beta function and anomalous dimensions are related. We will sketch a derivation of the latter in sect. 7.

The cancellation of UV divergences in the Wess-Zumino model was first analyzed in 2) shortly after the model was introduced. With the development of superfield & supergraph techniques in superspace it became possible to display all simplifications due to supersymmetry.³⁾⁻⁶⁾ A proof of the non-renormalization theorem was given in full details for the first time in 4), which relies on elaborate uses of the superfield techniques. The simplest proof is the one given by Seiberg,^{7),8)} where the coupling constants are promoted to chiral or antichiral constant superfields. Our elementary proof does not rely on superfields, but relies on the formulation of field theory using the exact renormalization group (ERG) differential equation.^{9),10)} Though there is nothing fancy about the ERG formalism, its familiarity cannot still be taken for granted, and we give sufficient background in sects. 3 & 4.

Our proof, to be given in sect. 5, is by no means the first given within the ERG formalism. The amalgamation of the superfield and ERG formalisms was obtained in 11) and 12), for which the result of 4) directly applies to proving the non-renormalization of the superpotential. A recent work by Rosten¹³⁾ also uses ERG to prove the non-renormalization theorem without using the result of 4). Since our proof closely resembles Rosten's (sect. V of 13)), we feel obliged to make comparisons in order to justify the publication of the present paper. In 13), a more general formulation of ERG is used to study theories of a scalar chiral superfield, not necessarily renormalizable perturbatively. Especially, the use of Fourier transforms in superspace makes his proof of non-renormalization less transparent to follow compared with our work. Our proof is simpler, depending only on the linearization of

the supersymmetry transformation by auxiliary fields, and it makes the advantage of the ERG formalism more explicit. It is a main goal of the present paper to show that ERG provides such a straightforward formulation of renormalizable theories.

We give a definite definition of the Wilson action S_Λ with momentum cutoff Λ . The non-renormalization theorem we prove applies only to S_Λ with strictly positive Λ . We use the four dimensional euclidean space throughout the paper.

§2. The classical action and its symmetry

The classical action of the Wess-Zumino model, without auxiliary fields, is given by

$$S'_{cl} = - \int d^4x \left[\partial_\mu \bar{\phi} \partial_\mu \phi + |m|^2 |\phi|^2 + \bar{\chi}_L \sigma \cdot \partial \chi_R + \frac{m}{2} \bar{\chi}_R \chi_R + \frac{\bar{m}}{2} \bar{\chi}_L \chi_L \right. \\ \left. + g \phi \frac{1}{2} \bar{\chi}_R \chi_R + \bar{g} \bar{\phi} \frac{1}{2} \bar{\chi}_L \chi_L + \frac{|g|^2}{4} |\phi|^4 + m \phi \frac{\bar{g}}{2} \bar{\phi}^2 + \bar{m} \bar{\phi} \frac{g}{2} \phi^2 \right], \quad (2.1)$$

where m is a complex mass parameter, and g is a complex dimensionless coupling. We use a bar to denote complex conjugation except for the right- or left-handed two-component spinors $\chi_{R,L}$, for which

$$\bar{\chi}_{R,L} \equiv \chi_{R,L}^T \sigma_y. \quad (2.2)$$

We also define

$$\sigma_\mu \equiv (\vec{\sigma}, i), \quad \bar{\sigma}_\mu \equiv (\vec{\sigma}, -i). \quad (\mu = 1, \dots, 4) \quad (2.3)$$

Shifting ϕ by $v \equiv \frac{m}{g}$, we obtain¹⁴⁾

$$S'_{cl} = - \int d^4x \left[\partial_\mu \bar{\phi} \partial_\mu \phi + \bar{\chi}_L \sigma \cdot \partial \chi_R \right. \\ \left. + g \phi \frac{1}{2} \bar{\chi}_R \chi_R + \bar{g} \bar{\phi} \frac{1}{2} \bar{\chi}_L \chi_L + \frac{|g|^2}{4} (\phi^2 - v^2) (\bar{\phi}^2 - \bar{v}^2) \right]. \quad (2.4)$$

With the shift, the action is invariant under the \mathbf{Z}_2 transformation:

$$\phi \rightarrow -\phi, \quad \bar{\phi} \rightarrow -\bar{\phi}, \quad \chi_R \rightarrow i\chi_R, \quad \chi_L \rightarrow -i\chi_L. \quad (2.5)$$

This symmetry is broken spontaneously by the non-vanishing VEV $\langle \phi \rangle = v$.

Using complex auxiliary fields F, \bar{F} , we obtain an equivalent action

$$S_{cl} \equiv S'_{cl} - \int d^4x \left\{ \bar{F} + i \frac{g}{2} (\phi^2 - v^2) \right\} \left\{ F + i \frac{\bar{g}}{2} (\bar{\phi}^2 - \bar{v}^2) \right\} \\ = - \int d^4x \left[\partial_\mu \bar{\phi} \partial_\mu \phi + \bar{\chi}_L \sigma \cdot \partial \chi_R + \bar{F} F \right. \\ \left. + \frac{g}{2} (\phi \bar{\chi}_R \chi_R + i F \phi^2) + \frac{\bar{g}}{2} (\bar{\phi} \bar{\chi}_L \chi_L + i \bar{F} \bar{\phi}^2) - i F \frac{g v^2}{2} - i \bar{F} \frac{\bar{g} \bar{v}^2}{2} \right]. \quad (2.6)$$

The unfamiliar factors of the imaginary i are due to our use of the euclidean metric. The action S_{cl} , written as above, depends on the dimensionless g , the squared mass parameter v^2 , and their complex conjugates.

The classical action is invariant under the following linear $N = 1$ supersymmetry transformation:

$$\begin{aligned}\delta\phi &= \bar{\chi}_R \xi_R, & \delta\bar{\phi} &= \bar{\chi}_L \xi_L, \\ \delta\chi_R &= \partial_\mu \phi \bar{\sigma}_\mu \xi_L - iF \xi_R, & \delta\chi_L &= \partial_\mu \bar{\phi} \sigma_\mu \xi_R - i\bar{F} \xi_L, \\ \delta F &= -i\partial_\mu \bar{\chi}_R \bar{\sigma}_\mu \xi_L, & \delta\bar{F} &= -i\partial_\mu \bar{\chi}_L \sigma_\mu \xi_R,\end{aligned}\tag{2.7}$$

where $\xi_{R,L}$ are anticommuting constant spinors. We note that the chiral fields (ϕ, χ_R, F) and the antichiral fields $(\bar{\phi}, \chi_L, \bar{F})$ do not mix with each other under the supersymmetry transformation.

The parameters g and v are both complex, but their phases are unphysical.⁸⁾ To see this, we consider the phase changes

$$\begin{aligned}g &\longrightarrow e^{i\alpha} g, & \bar{g} &\longrightarrow e^{-i\alpha} \bar{g}, \\ v &\longrightarrow e^{i\beta} v, & \bar{v} &\longrightarrow e^{-i\beta} \bar{v}.\end{aligned}\tag{2.8}$$

The action S_{cl} remains invariant, if we change the phases of the fields at the same time as follows:

$$\begin{aligned}\phi &\longrightarrow e^{i\beta} \phi, & \bar{\phi} &\longrightarrow e^{-i\beta} \bar{\phi}, \\ \chi_R &\longrightarrow e^{-i\frac{\alpha+\beta}{2}} \chi_R, & \chi_L &\longrightarrow e^{i\frac{\alpha+\beta}{2}} \chi_L, \\ F &\longrightarrow e^{-i(\alpha+2\beta)} F, & \bar{F} &\longrightarrow e^{i(\alpha+2\beta)} \bar{F}.\end{aligned}\tag{2.9}$$

The transformation (2.9) with $\alpha = 0$ is the R-symmetry transformation with R-character $\frac{1}{3}$, and that with $\alpha + 2\beta = 0$ is the R-symmetry transformation with R-character 1.⁶⁾ The cubic interaction terms are invariant under the former, and the terms linear in F, \bar{F} are invariant under the latter. With g and v both non-vanishing, (2.9) is not a symmetry transformation.

§3. Momentum cutoff

We regularize the theory using a smooth momentum cutoff. We split the bare action into the free and interaction parts:

$$S_B = S_{F,B} + S_{I,B}.\tag{3.1}$$

We impose that each of $S_{F,B}$ and $S_{I,B}$ be invariant under the supersymmetry transformation (2.7). The free part is given by

$$S_{F,B} \equiv - \int_p \frac{1}{K(p/\Lambda_0)} \left(p^2 \bar{\phi}(-p) \phi(p) + \bar{\chi}_L(-p) i\sigma \cdot p \chi_R(p) + \bar{F}(-p) F(p) \right),\tag{3.2}$$

where \int_p is a short-hand notation for $\int d^4p/(2\pi)^4$, and $K(p)$ is a smooth cutoff function satisfying the three properties:

1. $K(p)$ is a smooth non-negative decreasing function of p^2 .
2. $K(p)$ vanishes as $p^2 \rightarrow \infty$ fast enough for UV finiteness. (In our case, it suffices that K vanishes as fast as $\frac{1}{p^4}$.)
3. $K(p) = 1$ for $p^2 < 1$.

The free action implies the following propagators:

$$\langle \phi(p) \bar{\phi}(-p) \rangle_{S_{F,B}} = \frac{K(p/\Lambda_0)}{p^2}, \quad (3.3)$$

$$\langle \chi_R(p) \bar{\chi}_L(-p) \rangle_{S_{F,B}} = \frac{K(p/\Lambda_0)}{p^2} (-i) p \cdot \bar{\sigma}, \quad (3.4)$$

$$\langle F(p) \bar{F}(-p) \rangle_{S_{F,B}} = K(p/\Lambda_0). \quad (3.5)$$

Thanks to the cutoff function K , the propagation of high momentum modes are suppressed strongly, and the UV divergences are regularized.

The invariance under (2.7) and perturbative renormalizability constrain the interaction action $S_{I,B}$ to have the following form:

$$\begin{aligned} S_{I,B} = \int d^4x \Big[& z_2 \{ \partial_\mu \bar{\phi} \partial_\mu \phi + \bar{\chi}_L \sigma \cdot \partial \chi_R + \bar{F} F \} \\ & + (-1 + z_3) \left\{ \frac{g}{2} (\phi \bar{\chi}_R \chi_R + i F \phi^2) + \frac{\bar{g}}{2} (\bar{\phi} \bar{\chi}_L \chi_L + i \bar{F} \bar{\phi}^2) \right\} \\ & + \frac{g v^2}{2} i F + \frac{\bar{g} \bar{v}^2}{2} i \bar{F} \Big]. \end{aligned} \quad (3.6)$$

The renormalization constants z_2 & z_3 depend on the cutoff Λ_0 logarithmically. The invariance under the R-symmetry transformations (2.8) & (2.9) implies that both z_2 and z_3 are functions of $|g|^2$. The last two terms, linear in F or \bar{F} , give UV finite tadpoles, and they are not renormalized ^{*)}.

The non-renormalization theorem is that

$$z_3 = 0. \quad (3.7)$$

We wish to prove this using the ERG differential equation, which we will introduce in the next section.

§4. Wilson action

We renormalize the theory by examining the Wilson action with a non-vanishing but finite cutoff Λ , instead of examining the correlation functions. Roughly speaking, the Wilson action is obtained from the bare action with UV cutoff Λ_0 by integration over the momentum modes above the IR cutoff Λ . More precisely, the Wilson action is defined as follows:

$$S_\Lambda \equiv S_{F,\Lambda} + S_{I,\Lambda}, \quad (4.1)$$

where the free part is the same as $S_{F,B}$ except Λ_0 is replaced by Λ :

$$S_{F,\Lambda} \equiv - \int_p \frac{1}{K(p/\Lambda)} (p^2 \bar{\phi}(-p) \phi(p) + \bar{\chi}_L(-p) i \sigma \cdot p \chi_R(p) + \bar{F}(-p) F(p)). \quad (4.2)$$

^{*)} We may regard this as part of the non-renormalization theorem.

This is supersymmetric on its own. The interaction part is defined by

$$\begin{aligned} \exp [S_{I,\Lambda}[\phi, \dots, \bar{F}]] \equiv & \int [d\phi' d\bar{\phi}'] [d\chi'_R d\chi'_L] [dF' d\bar{F}'] \exp \left[\right. \\ & - \int_p \frac{1}{K(p/\Lambda_0) - K(p/\Lambda)} (p^2 \bar{\phi}'(-p) \phi'(p) + \bar{\chi}'_L(-p) i\sigma \cdot p \chi'_R(p) + \bar{F}'(-p) F'(p)) \\ & \left. + S_{I,B}[\phi + \phi', \dots, \bar{F} + \bar{F}'] \right]. \end{aligned} \quad (4.3)$$

In terms of Feynman graphs, $S_{I,\Lambda}$ consists of connected graphs with the elementary three-point vertices and the propagators multiplied by the cutoff function $K(p/\Lambda_0) - K(p/\Lambda)$, which is approximately 1 for $\Lambda^2 < p^2 < \Lambda_0^2$, and zero for $p^2 < \Lambda^2$. Note that $S_{I,\Lambda}$ is not only supersymmetric, but also invariant under the R-symmetry transformations (2.8) & (2.9).

Alternatively, we can define $S_{I,\Lambda}$ by the ERG differential equation

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} S_{I,\Lambda} = & \int_q \frac{\Delta(q/\Lambda)}{q^2} \left[\frac{\delta S_{I,\Lambda}}{\delta \phi(q)} \frac{\delta S_{I,\Lambda}}{\delta \bar{\phi}(-q)} + \frac{\delta^2 S_{I,\Lambda}}{\delta \phi(q) \delta \bar{\phi}(-q)} \right. \\ & + S_{I,\Lambda} \overleftarrow{\delta} \frac{(-i)q \cdot \bar{\sigma}}{\delta \chi_R(q)} \overrightarrow{\delta} \frac{S_{I,\Lambda}}{\delta \bar{\chi}_L(-q)} - \text{Tr} \frac{\overrightarrow{\delta}}{\delta \bar{\chi}_L(-q)} S_{I,\Lambda} \overleftarrow{\delta} \frac{(-i)q \cdot \bar{\sigma}}{\delta \chi_R(q)} \\ & \left. + q^2 \frac{\delta S_{I,\Lambda}}{\delta F(q)} \frac{\delta S_{I,\Lambda}}{\delta \bar{F}(-q)} + q^2 \frac{\delta^2 S_{I,\Lambda}}{\delta F(q) \delta \bar{F}(-q)} \right], \end{aligned} \quad (4.4)$$

where

$$\Delta(q) \equiv -2q^2 \frac{d}{dq^2} K(q), \quad (4.5)$$

and by the initial condition

$$S_{I,\Lambda} \Big|_{\Lambda=\Lambda_0} = S_{I,B}. \quad (4.6)$$

Note that $\Delta(q) = 0$ for $q^2 < 1$; hence, the q in the ERG differential equation is integrated over $q^2 > \Lambda^2$. We can regard (4.3) as the integral formula of (4.4) & (4.6).

The dependence of $S_{I,\Lambda}$ on the squared mass parameter v^2 and its complex conjugate can be obtained trivially. For this, we note that if $S_{I,\Lambda}$ is a solution of (4.4), so is

$$\tilde{S}_{I,\Lambda} \equiv S_{I,\Lambda} - i \frac{gv^2}{2} F(0) - i \frac{\bar{g}\bar{v}^2}{2} \bar{F}(0), \quad (4.7)$$

where

$$F(0) \equiv \int d^4x F(x), \quad \bar{F}(0) \equiv \int d^4x \bar{F}(x). \quad (4.8)$$

Hence, $\tilde{S}_{I,\Lambda}$ is the Wilson action of the massless theory, corresponding to $v^2 = 0$.

Let us now expand $\tilde{S}_{I,\Lambda}$ in powers of fields. The supersymmetry and the invariance under the R-symmetry transformations (2.8) & (2.9) imply

$$\begin{aligned} \tilde{S}_{I,\Lambda} = & \int_p \mathcal{V}_2(p) \{ p^2 \bar{\phi}(-p) \phi(p) + \bar{\chi}_L(-p) i \sigma \cdot p \chi_R(p) + \bar{F}(-p) F(p) \} \\ & + \int_{p_1, p_2, p_3} \mathcal{V}_3(p_1, p_2, p_3) (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3) \\ & \times \left\{ \frac{g}{2} (\phi(p_1) \bar{\chi}_R(p_2) \chi_R(p_3) + i F(p_1) \phi(p_2) \phi(p_3)) \right. \\ & \left. + \frac{\bar{g}}{2} (\bar{\phi}(p_1) \bar{\chi}_L(p_2) \chi_L(p_3) + i \bar{F}(p_1) \bar{\phi}(p_2) \bar{\phi}(p_3)) \right\} + \dots \end{aligned} \quad (4.9)$$

up to terms cubic in the fields. The coefficients \mathcal{V}_2 and \mathcal{V}_3 are scalar functions dependent on $|g|^2$. \mathcal{V}_3 is symmetric with respect to the three momenta.

Expanding $\mathcal{V}_2(p)$ in powers of $\frac{p^2}{\Lambda^2}$, we obtain

$$\mathcal{V}_2(p) = c_2(\ln \Lambda/\mu) + \dots \quad (4.10)$$

Similarly, expanding $\mathcal{V}_3(p_1, p_2, p_3)$, we obtain

$$\mathcal{V}_3(p_1, p_2, p_3) = -1 + c_3(\ln \Lambda/\mu) + \dots \quad (4.11)$$

Both c_2 and c_3 are momentum independent constants, but they depend on the cutoff Λ logarithmically. We have introduced an arbitrary momentum scale μ to make the argument of the log dimensionless. The initial condition (4.6) implies

$$c_2(\ln \Lambda_0/\mu) = z_2, \quad c_3(\ln \Lambda_0/\mu) = z_3. \quad (4.12)$$

Now, what about renormalization? We tune the renormalization constants z_2, z_3 so that the limit of the Wilson action

$$\lim_{\Lambda_0 \rightarrow \infty} S_{I,\Lambda} \quad (4.13)$$

exists order by order in $|g|^2$. More concretely, we choose z_2, z_3 such that

$$c_2(0) = c_3(0) = 0 \quad (4.14)$$

are satisfied. This is a particular renormalization condition, where μ plays the role of a renormalization scale. To prove the non-renormalization theorem (3.7), it suffices to show

$$- \Lambda \frac{\partial}{\partial \Lambda} c_3(\ln \Lambda/\mu) = 0. \quad (4.15)$$

Then, (4.12) and (4.14) imply

$$c_3(\ln \Lambda/\mu) = 0. \quad (4.16)$$

This is an alternative form of the non-renormalization theorem (3.7).

§5. Proof of the non-renormalization theorem

To prove (4.15), we examine the part of the Wilson action proportional to

$$g (iF(0)\phi(0)\phi(0) + \phi(0)\bar{\chi}_R(0)\chi_R(0)), \quad (5.1)$$

where all the fields are evaluated at zero momentum. Its coefficient is $-1+c_3(\ln \Lambda/\mu)$. The ERG differential equation (4.4) gives

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} c_3(\ln \Lambda/\mu) \cdot g (iF(0)\phi(0)\phi(0) + \phi(0)\bar{\chi}_R(0)\chi_R(0)) &= \int_q \frac{\Delta(q/\Lambda)}{q^2} \\ &\times \left\{ \frac{\delta^2}{\delta\phi(q)\delta\bar{\phi}(-q)} \mathcal{V}_5 - \text{Tr} \frac{\vec{\delta}}{\delta\bar{\chi}_L(-q)} \mathcal{V}_5 \frac{\overleftarrow{\delta}}{\delta\chi_R(q)} (-i)q \cdot \bar{\sigma} + q^2 \frac{\delta^2}{\delta\bar{F}(-q)\delta F(q)} \mathcal{V}_5 \right\}, \end{aligned} \quad (5.2)$$

where \mathcal{V}_5 is the part of $S_{I,\Lambda}$ proportional to one chiral field with momentum q , one antichiral field with momentum $-q$, and three chiral fields with zero momentum. Since $\Delta(q/\Lambda) = 0$ for $q^2 < \Lambda^2$, we can restrict $q^2 > \Lambda^2$ in \mathcal{V}_5 . We enumerate all possibilities for \mathcal{V}_5 that satisfy the constraints of supersymmetry and the invariance under the R-symmetry transformations (2.8) & (2.9). The most general \mathcal{V}_5 is given in the following form:

$$\begin{aligned} \mathcal{V}_5 &= g \int_q [\{ \bar{\chi}_R(-q)(-i)q \cdot \bar{\sigma} \chi_L(q) + iF(-q)i\bar{F}(q) - q^2 \bar{\phi}(q)\phi(-q) \} \\ &\quad \times \frac{1}{2} \{ w_1(q)\phi(0)\phi(0)iF(0) + w_2(q)\bar{\chi}_R(0)\chi_R(0)\phi(0) \} \\ &\quad + (w_1(q) - w_2(q)) iF(0) \{ \phi(0) (\bar{\chi}_R(0)(-i)q \cdot \bar{\sigma} \chi_L(q) + iF(0)i\bar{F}(q)) \phi(-q) \\ &\quad + \left(\phi(-q)\frac{1}{2}\bar{\chi}_R(0)\chi_R(0) + \phi(0)\bar{\chi}_R(-q)\chi_R(0) \right) i\bar{F}(q) \}] , \end{aligned} \quad (5.3)$$

where $w_1(q), w_2(q)$ are arbitrary functions of q^2 and $|g|^2$. For example, at 1-loop, we obtain

$$\begin{cases} w_1(q) &= 2|g|^4 \frac{1-K(q/\Lambda)}{q^2} \int_r \frac{(1-K((r+q)/\Lambda))^2 (1-K(r/\Lambda))^2}{r^4 (r+q)^4} r^2, \\ w_2(q) &= |g|^4 \frac{1-K(q/\Lambda)}{q^2} \int_r \frac{(1-K((r+q)/\Lambda))^2 (1-K(r/\Lambda))^2}{r^4 (r+q)^4} \left(r^2 - \frac{q^2}{2} \right). \end{cases} \quad (5.4)$$

Substituting (5.3) into (5.2), we can verify (4.15) by simple algebra. Please see Appendix A for an explanation of this vanishing and a more systematic derivation of (5.3).

§6. Generalization

The non-renormalization theorem (3.7) or (4.16) can be generalized further using the invariance of the Wilson action under the R-symmetry transformations (2.8) & (2.9).

Let us first define the superpotential $V_\Lambda[\Phi] + \bar{V}_\Lambda[\bar{\Phi}]$ using the Wilson action. The superpotential $V_\Lambda[\Phi]$ is the part of $S_{I,\Lambda}$ that depends only on the chiral fields

$\Phi \equiv \{\phi, \chi_R, F\}$ with no space derivatives. Similarly, $\bar{V}_A[\bar{\Phi}]$ is the part dependent only on $\bar{\Phi} \equiv \{\bar{\phi}, \chi_L, \bar{F}\}$ with no space derivatives. \bar{V}_A is basically the complex conjugate of V , where we interpret χ_L as the complex conjugate of χ_R . The non-renormalization theorem (4.16) implies that the cubic part of $V_A[\Phi]$ is given exactly by

$$-\frac{g}{2} \int d^4x \{iF\phi^2 + \phi\bar{\chi}_R\chi_R\}. \quad (6.1)$$

Now, what about higher order terms?

We know that the higher order terms depend only on the chiral fields (ϕ, χ_R, F) and the coupling constants g, \bar{g} . The most general supersymmetric term consisting of $n(\geq 4)$ chiral fields is given by

$$g|g|^{2l} \int d^4x \left(iF\phi^{n-1} + \frac{n-1}{2}\phi^{n-2}\bar{\chi}_R\chi_R \right),$$

where l is an arbitrary non-negative integer. The first factor of g is for the invariance under the R-symmetry transformations (2.8) & (2.9) with $\beta = 0$. But the above is not invariant under the R-symmetry transformations (2.8) and (2.9) with $\beta \neq 0$.

Thus, we conclude that the chiral part of the superpotential is exactly given by

$$V_A[\Phi] = -\frac{g}{2} \int d^4x \{iF(\phi^2 - v^2) + \phi\bar{\chi}_R\chi_R\}. \quad (6.2)$$

§7. Implications

For completeness of the paper, we would like to give an argument that relates the anomalous dimensions of the fields and v^2 to the beta function of $|g|^2$. The anomalous dimensions beyond 1-loop and the beta function beyond 2-loop are scheme dependent, and the relations we derive are valid only for the particular scheme adopted in the present paper. The anomalous dimensions and beta function can be derived from the dependence of the Wilson action S_A on the renormalization point μ as discussed in 15) and 16). In the following we give a hand-waving but more straightforward derivation, relegating a systematic derivation to Appendix B.

We replace the bare action (3.1) naïvely by

$$\begin{aligned} S_B = \int d^4x & \left[-Z \{ \partial_\mu \bar{\phi} \partial_\mu \phi + \bar{\chi}_L \sigma \cdot \partial \chi_R + \bar{F} F \} \right. \\ & - Z^{\frac{3}{2}} Z_g (g \{ iF\phi^2 + \phi\bar{\chi}_R\chi_R \} + \bar{g} \{ i\bar{F}\bar{\phi}^2 + \bar{\phi}\bar{\chi}_L\chi_L \}) \\ & \left. + Z^{\frac{1}{2}} Z_g Z_{v^2} \left\{ \frac{gv^2}{2} iF + \frac{\bar{g}v^2}{2} i\bar{F} \right\} \right], \end{aligned} \quad (7.1)$$

where

$$Z = 1 - c_2, \quad Z^{\frac{3}{2}} Z_g = 1 - c_3, \quad Z^{\frac{1}{2}} Z_g Z_{v^2} = 1. \quad (7.2)$$

Z is the renormalization constant for the fields, $g_B = Z_g g$ is the bare coupling, and $v_B^2 = Z_{v^2} v^2$ is the bare squared mass parameter. The non-renormalization theorem

(3·7) gives

$$Z = Z_g^{-\frac{2}{3}}, \quad Z_{v^2} = Z. \quad (7.3)$$

Defining the beta function β , the anomalous dimension β_{v^2} of v^2 , and that of fields γ by

$$\mu \frac{\partial}{\partial \mu} \ln Z_g = \frac{\beta}{2|g|^2}, \quad \mu \frac{\partial}{\partial \mu} \ln Z_{v^2} = \beta_{v^2}, \quad \mu \frac{\partial}{\partial \mu} \ln Z = 2\gamma, \quad (7.4)$$

we obtain

$$\frac{\beta}{2|g|^2} = -3\gamma, \quad \beta_{v^2} = 2\gamma, \quad (7.5)$$

as first found in 2).

§8. Concluding remarks

In this paper we have proven the non-renormalization theorem (4·16) & (6·2) by an elementary use of ERG. Even though our proof does not rely on the superfield techniques, it still depends on the linearization of the supersymmetry transformation via auxiliary fields. The Wess-Zumino model can be constructed without auxiliary fields ^{*)}, and it should be interesting to formulate and prove the non-renormalization theorem without auxiliary fields.

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Appendix A

—— Non-renormalization theorem in terms of component fields ——

In the component field formalism, the algebraic source of the non-renormalization theorems of supersymmetric theories is the cohomological structure of the supersymmetry transformation: any supersymmetric invariants can be written as multiple supervariations of integrals over field polynomials.^{17),18)}

Let us write the N=1 supersymmetry transformation δ in terms of left and right supersymmetry generators:

$$\delta = \bar{\xi}_L Q_L + \bar{\xi}_R Q_R. \quad (A.1)$$

Any supersymmetric invariant in four dimensions is the highest component of the respective super multiplet, and it can be written as

$$I = \bar{Q}_R Q_R \bar{Q}_L Q_L X + \bar{Q}_R Q_R Y + \bar{Q}_L Q_L \bar{Y}, \quad (A.2)$$

where X , Y , and \bar{Y} are integrals over polynomials of fields that belong to the respective supersymmetry multiplets. Y satisfies $Q_L Y = 0$, and \bar{Y} satisfies $Q_R \bar{Y} = 0$. But X need not to be invariant under Q_L or Q_R .

^{*)} For a construction within the ERG formalism, see 20).

For instance, let us construct the chiral part, $\bar{Q}_R Q_R Y$, for the Wilson action of the Wess-Zumino model. By analyzing the supersymmetry transformations of the fields, one finds that Y can only be written as (an integral of) ϕ^n . All the other polynomials annihilated by Q_L can be written as $\bar{Q}_L Q_L \tilde{X}'$ so that these make part of the first term in (A.2). For instance, $\partial^2 \phi \cdot \phi^{n-1}$ can be written as $\bar{Q}_L Q_L (F \phi^{n-1})$, and $\phi^n \bar{F}^m$ as $\bar{Q}_L Q_L (\phi^n \bar{\phi} \bar{F}^{m-1})$.

It is now straightforward to rewrite Seiberg's proof of the non-renormalization theorem^{7),8)} using component fields. Following Seiberg,^{7),8)} we promote the parameters g, \bar{g} to constant fields that belong to chiral multiplets with the R-characters determined by (2.8). Then, a similar argument as above shows that Y is a polynomial of only g and ϕ . Since the action is R-symmetric, Y must transform as $e^{i(\alpha+3\beta)}$ under (2.8) & (2.9). This leaves only a constant multiple of $\int d^4x g \phi^3$. Hence, the chiral (antichiral) part of the action related with Y (\bar{Y}) does not receive radiative corrections. This is exactly the same argument as Seiberg's.^{7),8)} Therefore, we conclude that any radiative correction to the supersymmetric Wilson action can be written as $\bar{Q}_R Q_R \bar{Q}_L Q_L X$.

Now, using the above construction of supersymmetric invariants, we can easily derive \mathcal{V}_5 given by (5.3). Note that the quintic vertex \mathcal{V}_5 consists of terms of the form

$$\Phi(q) \bar{\Phi}(-q) \cdot \Phi(0) \Phi(0) \Phi(0), \quad (\text{A.3})$$

where Φ and $\bar{\Phi}$ denote generic chiral and antichiral component fields, and the mass dimension of $\Phi(0) \Phi(0) \Phi(0)$ must be at most 5. Considering the R-symmetry (2.8) & (2.9) and the dimensionality of the fields, we conclude the following:

1. There is no possible Y satisfying $Q_L Y = 0$ that cannot be written as $\bar{Q}_L Q_L X$.
2. There are nine possible field polynomials for X , but under the action of

$$\bar{Q}_L Q_L \bar{Q}_R Q_R,$$

only the following two are independent:

$$X = \{g \bar{\phi} \phi \cdot \phi \bar{\chi}_R \chi_R, g \bar{\phi} \phi \cdot i F \phi^2\}.$$

It is then easy to show that (5.3) can be obtained as

$$\mathcal{V}_5 = \frac{g}{8} \bar{Q}_R Q_R \bar{Q}_L Q_L \int_q [w_1(q) \bar{\phi}(q) \phi(-q) i F(0) \phi(0) \phi(0) + w_2(q) \bar{\phi}(q) \phi(-q) \phi(0) \bar{\chi}_R(0) \chi_R(0)]. \quad (\text{A.4})$$

Note that in Sec.V, \mathcal{V}_5 is found by enumerating all the possibilities allowed by R-symmetry and dimensionality. However, the above construction is more systematic and simpler. Moreover, (4.15) comes as no surprise since the loop contraction

$$\int_q \frac{\Delta(q/\Lambda)}{q^2} \left(\frac{\delta^2}{\delta \phi(q) \delta \phi(-q)} + \cdots \right) \quad (\text{A.5})$$

is supersymmetric and commutes with $\bar{Q}_R Q_R \bar{Q}_L Q_L$. Hence, the loop contraction gives $\bar{Q}_R Q_R \bar{Q}_L Q_L i F(0) \phi(0) \phi(0)$ and $\bar{Q}_R Q_R \bar{Q}_L Q_L \phi(0) \bar{\chi}_R(0) \chi_R(0)$, both of which vanish identically.

We wish to give more details elsewhere on the use of the component field formalism for the cohomological construction and for the proof of the non-renormalization theorem.

Appendix B

— The relation among β , γ , and β_{v^2} —

The derivation of the beta function and anomalous dimensions from the Wilson action has been discussed in 15) and 16). Here, we follow the latter.

The μ dependence of the Wilson action can be written as

$$-\mu \frac{\partial}{\partial \mu} S_\Lambda = \frac{\beta}{2|g|^2} (g\mathcal{O}_g + \bar{g}\mathcal{O}_{\bar{g}}) + \beta_{v^2} (v^2\mathcal{O}_{v^2} + \bar{v}^2\mathcal{O}_{\bar{v}^2}) + \gamma\mathcal{N}, \quad (\text{B}\cdot 1)$$

where β , β_{v^2} , and γ are functions of $|g|^2$, and the composite operators \mathcal{O}_g , \mathcal{O}_{v^2} , \mathcal{N} are defined by

$$\mathcal{O}_g \equiv -\partial_g S_{I,\Lambda}, \quad (\text{B}\cdot 2)$$

$$\mathcal{O}_{v^2} \equiv -\partial_{v^2} S_{I,\Lambda} = -\frac{i}{2}gF(0), \quad (\text{B}\cdot 3)$$

$$\begin{aligned} \mathcal{N} \equiv & -\int_p K(p/\Lambda) \left[[\phi](p) \frac{\delta S_\Lambda}{\delta \phi(p)} + \frac{\delta}{\delta \phi(p)} [\phi](p) + [\bar{\phi}](p) \frac{\delta S_\Lambda}{\delta \bar{\phi}(p)} + \frac{\delta}{\delta \bar{\phi}(p)} [\bar{\phi}](p) \right. \\ & + S_\Lambda \frac{\overleftarrow{\delta}}{\delta \chi_R(p)} [\chi_R](p) - \text{Tr} [\chi_R](p) \frac{\overleftarrow{\delta}}{\delta \chi_R(p)} + S_\Lambda \frac{\overleftarrow{\delta}}{\delta \chi_L(p)} [\chi_L](p) - \text{Tr} [\chi_L](p) \frac{\overleftarrow{\delta}}{\delta \chi_L(p)} \\ & \left. + [F](p) \frac{\delta S_\Lambda}{\delta F(p)} + \frac{\delta}{\delta F(p)} [F](p) + [\bar{F}](p) \frac{\delta S_\Lambda}{\delta \bar{F}(p)} + \frac{\delta}{\delta \bar{F}(p)} [\bar{F}](p) \right], \end{aligned} \quad (\text{B}\cdot 4)$$

where the elementary fields in square brackets are defined by

$$[\phi](p) \equiv \phi(p) + \frac{1 - K(p/\Lambda)}{p^2} \frac{\delta S_{I,\Lambda}}{\delta \phi(-p)}, \dots \quad (\text{B}\cdot 5)$$

The operator \mathcal{N} is the equation of motion operator that counts the number of fields:

$$\left\langle \mathcal{N} \underbrace{\phi(p_1) \cdots}_{N \text{ fields}} \right\rangle = N \langle \phi(p_1) \cdots \rangle. \quad (\text{B}\cdot 6)$$

β is the beta function of $|g|^2$, β_{v^2} is the anomalous dimension of v^2 , and γ is the common anomalous dimension of the matter fields, since (B·1) implies

$$\left(-\mu \frac{\partial}{\partial \mu} + \frac{\beta}{2|g|^2} (g\partial_g + \bar{g}\partial_{\bar{g}}) + \beta_{v^2} (v^2\partial_{v^2} + \bar{v}^2\partial_{\bar{v}^2}) \right) \langle \phi(p_1) \cdots \rangle = N\gamma \langle \phi(p_1) \cdots \rangle. \quad (\text{B}\cdot 7)$$

To extract the coefficients $\beta, \beta_{v^2}, \gamma$, we expand the Wilson action in powers of fields, and expand the coefficients in powers of momenta. Alternatively, we can examine the asymptotic behavior of the Wilson action for large Λ :¹⁶⁾

1. The asymptotic behavior of $-\mu\partial_\mu S_\Lambda$ is given by

$$-\mu\frac{\partial}{\partial\mu}S_\Lambda \xrightarrow{\Lambda\rightarrow\infty} \Lambda\frac{\partial}{\partial\Lambda}c_2(\ln\Lambda/\mu) \int d^4x \left(\partial_\mu\bar{\phi}\partial_\mu\phi + \bar{\chi}_L\sigma\cdot\partial\chi_R + \bar{F}F \right). \quad (\text{B}\cdot 8)$$

2. The asymptotic behavior of the g, \bar{g} derivatives is given by

$$\begin{aligned} g\mathcal{O}_g + \bar{g}\mathcal{O}_{\bar{g}} &= -g\partial_g S_\Lambda - \bar{g}\partial_{\bar{g}} S_\Lambda \\ &\xrightarrow{\Lambda\rightarrow\infty} \int d^4x \left[-2|g|^2 \frac{\partial}{\partial|g|^2} c_2(\ln\Lambda/\mu) \left(\partial_\mu\bar{\phi}\partial_\mu\phi + \bar{\chi}_L\sigma\cdot\partial\chi_R + \bar{F}F \right) \right. \\ &\quad + \frac{1}{2} \left(g\phi\bar{\chi}_R\chi_R + g\phi^2 iF + \bar{g}\bar{\phi}\bar{\chi}_L\chi_L + \bar{g}\bar{\phi}^2 i\bar{F} \right) \\ &\quad \left. - \frac{1}{2} \left(gv^2 iF + \bar{g}\bar{v}^2 i\bar{F} \right) \right]. \end{aligned} \quad (\text{B}\cdot 9)$$

3. \mathcal{O}_{v^2} and $\mathcal{O}_{\bar{v}^2}$ are exactly given by

$$v^2\mathcal{O}_{v^2} + \bar{v}^2\mathcal{O}_{\bar{v}^2} = -\frac{i}{2} \left(gv^2 F(0) + \bar{g}\bar{v}^2 \bar{F}(0) \right). \quad (\text{B}\cdot 10)$$

4. The asymptotic behavior of the field counting operator is the most complicated:

$$\begin{aligned} \mathcal{N} &\xrightarrow{\Lambda\rightarrow\infty} \int d^4x \left[2(1-c_2) \left(\partial_\mu\bar{\phi}\partial_\mu\phi + \bar{\chi}_L\sigma\cdot\partial\chi_R + \bar{F}F \right) \right. \\ &\quad + \frac{3}{2} \left(g\phi\bar{\chi}_R\chi_R + giF\phi^2 + \bar{g}\bar{\phi}\bar{\chi}_L\chi_L + \bar{g}i\bar{F}\bar{\phi}^2 \right) \\ &\quad \left. - \frac{1}{2} \left(gv^2 iF + \bar{g}\bar{v}^2 i\bar{F} \right) \right] \\ &\quad - 2 \int_p \frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2} \left(\frac{\delta^2 S_{I,\Lambda}}{\delta\phi(p)\delta\bar{\phi}(-p)} \right. \\ &\quad \left. + \text{Tr}(-i)p\cdot\bar{\sigma} \frac{\overrightarrow{\delta}}{\delta\bar{\chi}_L(-p)} S_{I,\Lambda} \frac{\overleftarrow{\delta}}{\delta\chi_R(p)} + p^2 \frac{\delta^2 S_{I,\Lambda}}{\delta\bar{F}(-p)\delta F(p)} \right), \end{aligned} \quad (\text{B}\cdot 11)$$

where the loop momentum p is of order Λ . For the loop integral, we need the part of $S_{I,\Lambda}$ including a chiral field of momentum p , an antichiral field of momentum $-p$, and a number of fields with momenta low compared with Λ . The part including three chiral fields at zero momentum vanishes as explained in the proof of the non-renormalization theorem. Only the part proportional to the kinetic term survives the loop integral.

Hence, we obtain

$$\begin{aligned} &\frac{\beta}{2|g|^2} (g\mathcal{O}_g + \bar{g}\mathcal{O}_{\bar{g}}) + \beta_{v^2} (v^2\mathcal{O}_{v^2} + \bar{v}^2\mathcal{O}_{\bar{v}^2}) + \gamma\mathcal{N} \\ &\xrightarrow{\Lambda\rightarrow\infty} \int d^4x \left[\left(-\beta \frac{\partial c_2}{\partial|g|^2} + 2\gamma(1-c_2) - 2\gamma \int_p \frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2} B_4(0,p) \right) \right. \\ &\quad \left. \cdot \left(\partial_\mu\bar{\phi}\partial_\mu\phi + \bar{\chi}_L\sigma\cdot\partial\chi_R + \bar{F}F \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{\beta}{2|g|^2} + 3\gamma \right) \cdot \frac{1}{2} (g\phi\bar{\chi}_R\chi_R + giF\phi^2 + \bar{g}\bar{\phi}\bar{\chi}_L\chi_L + \bar{g}i\bar{F}\bar{\phi}^2) \\
& + \left(-\frac{\beta}{2|g|^2} - \beta_{v^2} - \gamma \right) \frac{1}{2} (gv^2iF + \bar{g}\bar{v}^2i\bar{F}) \Big], \tag{B.12}
\end{aligned}$$

where $B(0, p)$ is the coefficient of a term in $S_{I, \Lambda}$, containing a pair of chiral and antichiral fields with momentum of order Λ and a pair with low momentum. Thus, we obtain the desired relation

$$\frac{\beta}{2|g|^2} = -3\gamma, \quad \beta_{v^2} = 2\gamma, \tag{B.13}$$

and an equation that determines the anomalous dimension γ :

$$\frac{\partial c_2(\ln \Lambda/\mu)}{\partial \ln \Lambda/\mu} = 6\gamma|g|^2 \frac{\partial c_2}{\partial |g|^2} + 2\gamma(1 - c_2) - 2\gamma \int_p \frac{K(p/\Lambda)(1 - K(p/\Lambda))}{p^2} B_4(0, p). \tag{B.14}$$

We can calculate γ by computing c_2 and B_4 perturbatively.

Up to 2-loop, we obtain

$$\begin{cases} B_4^{(0)}(0, p) &= -|g|^2 \frac{1-K(p/\Lambda)}{p^2}, \\ c_2^{(1)}(\ln \Lambda/\mu) &= \frac{|g|^2}{(4\pi)^2} \ln \frac{\Lambda}{\mu}, \\ c_2^{(2)}(\ln \Lambda/\mu) &= \frac{|g|^4}{(4\pi)^4} \left[\left(\ln \frac{\Lambda}{\mu} \right)^2 - a \ln \frac{\Lambda}{\mu} \right], \end{cases} \tag{B.15}$$

where the superscript denotes the number of loops, and the constant a is given by

$$\begin{aligned}
a \equiv (4\pi)^4 & \left[\frac{3}{2} \int_q \frac{\Delta(q)(1-K(q))^2}{q^4} \int_r \frac{1-K(r)}{r^2} \left(\frac{1-K(r+q)}{(r+q)^2} - \frac{1-K(r)}{r^2} \right) \right. \\
& \left. + \int_q \frac{\Delta(q)}{q^2} \int_r \frac{(1-K(r))^3(1-K(r+q))}{r^4(r+q)^2} \right]. \tag{B.16}
\end{aligned}$$

Note that the value of a depends on the choice of the cutoff function K .

Using the above, we obtain the following results for the anomalous dimension:

$$2\gamma^{(1)} = \dot{c}_2^{(1)}(\ln \Lambda/\mu) = |g|^2 \frac{1}{(4\pi)^2}, \tag{B.17}$$

$$\begin{aligned}
2\gamma^{(2)} &= \dot{c}_2^{(2)}(\ln \Lambda/\mu) - 4\gamma^{(1)} \cdot c_2^{(1)} - 2\gamma^{(1)}|g|^2 \int_q \frac{K(1-K)^2}{q^4} \\
&= -\frac{|g|^4}{(4\pi)^4} b, \tag{B.18}
\end{aligned}$$

where

$$b \equiv a + (4\pi)^2 \int_q \frac{K(1-K)^2}{q^4}$$

$$\begin{aligned}
&= (4\pi)^4 \left[\frac{3}{2} \int_q \frac{\Delta(q)(1-K(q))^2}{q^4} \int_r \frac{1-K(r)}{r^2} \left(\frac{1-K(r+q)}{(r+q)^2} - \frac{1-K(r)}{r^2} \right) \right. \\
&\quad \left. + \int_q \frac{\Delta(q)}{q^2} \int_r \frac{(1-K(r))^3(1-K(r+q))}{r^4(r+q)^2} + \frac{1}{(4\pi)^2} \int_q \frac{K(1-K)^2}{q^4} \right] \\
&= 1.
\end{aligned} \tag{B.19}$$

The value of b is independent of the choice of K^*).

Thus, up to 2-loop, we obtain

$$2\gamma = \frac{|g|^2}{(4\pi)^2} - \frac{|g|^4}{(4\pi)^4}. \tag{B.20}$$

This agrees with the known result.^{11),19)}

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^{*)} We have computed b using the sharp cutoff function $K(q) = \theta(1 - q^2)$.